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Characteristic cycles of local cohomology modules of monomial ideals

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Abstract

We study, by using the theory of algebraic $\mathscr{D}$-modules, the local cohomology modules supported on a monomial ideal I of the local regular ring $R=k[[x_1,\dots,x_n]]$, where k is a field of characteristic zero. We compute the characteristic cycle of $H_I^r(R)$ and $H_m^p(H_I^r(R))$, where m is the maximal ideal of R and I is a squarefree monomial ideal. As a consequence, we can decide when the local cohomology module $H_I^r(R)$ vanishes and compute the cohomological dimension $cd(R,I)$ in terms of the minimal primary decomposition of the monomial ideal I. We also give a Cohen–Macaulayness criterion for the local ring R/I and compute the Lyubeznik numbers $\lambda_{p,i}(R/I)=dim_k\, Ext_R^p(k,H_I^{n-i}(R))$. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In his paper [12] Lyubeznik uses the theory of algebraic $\mathscr{D}$-modules to study local cohomology modules. He proves, in particular, that if R is any regular ring containing a field of characteristic zero and $I\subseteq R$ is an ideal, the local cohomology modules $H_I^i(R)$ have the following properties:

(i) $H_m^j(H_I^i(R))$ is injective, where m is any maximal ideal of R.

(ii) $inj.dim_R\,(H_I^i(R)\le dim_R\, H_I^i(R)$.

(iii) The set of the associated primes of $H_I^i(R)$ is finite.

(iv) All the Bass numbers of $H_I^i(R)$ are finite.

By using the Frobenius map, the same results have been obtained by Huneke and Sharp [9] for regular rings containing a field of positive characteristic. By (iv),

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Lyubeznik defines a new set of numerical invariants for any local ring A containing a field, denoted by $\lambda_{p,i}(A)$. Namely, let $(R,\mathfrak{m},k)$ be a regular local ring of dimension n containing k, and A a local ring which admits a surjective ring homomorphism $\pi : R \to A$. Set $I = Ker\,\pi$. Then, $\lambda_{p,i}(A)$ is defined as the Bass number $\mu_p(\mathfrak{m}, H_I^{n-i}(R)) = dim_k\, Ext_R^p(k, H_I^{n-i}(R))$. This invariant depends only on A, i and p, but neither on R nor on π. Completion does not change $\lambda_{p,i}(A)$ so if A contains a field but it is not necessarily the quotient of a regular local ring then one can define $\lambda_{p,i}(A) = \lambda_{p,i}(\hat{A})$ where $\hat{A}$ is the completion of A with respect to the maximal ideal. Therefore, one can always assume $R = k[[x_1,\dots,x_n]]$, with $x_1,\dots,x_n$ independent variables.

In this paper we want to study the local cohomology modules of R supported on a monomial ideal $I \subseteq R$, where $R = k[x_1,\dots,x_n]$ or $R = k[[x_1,\dots,x_n]]$ and k is a field of characteristic zero. Since $H_I^i(R) = H^i_{\sqrt{I}}(R)$ we may assume I is reduced. It is well known that reduced monomial ideals have a minimal primary decompositon $I = I_1 \cap \cdots \cap I_m$, where the prime ideals $I_i \subseteq R$ have the form $(x_{i_1},\dots,x_{i_h})$, with $x_{i_j} \in \{x_1,\dots,x_n\}$. Ideals having this form are usually called face ideals. Local cohomology modules supported on a monomial ideal have been studied by Lyubeznik [11].

The local cohomology modules $H_I^r(R)$ have a natural structure of finitely generated $D(R,k)$-module, where $D(R,k)$ is the ring of k-linear differential operators of R. In particular, one can define the dimension and the multiplicity of a $D(R,k)$-module. By Bernstein's inequality the dimension of a non-zero finitely generated $D(R,k)$-module is greater or equal to n. The class of $D(R,k)$-modules of dimension n are called holonomic modules and form a category with good properties. The ring R, all the localizations of R by any element of R, as well as the local cohomology modules $H_J^r(R)$ for any ideal J are holonomic. In Section 2 we summarize all these results and recall that to any holonomic module M one can attach an invariant, the characteristic cycle, that allows us to get the support of M. We describe in detail the characteristic cycle of R and its localizations by a monomial.

In Section 3 we begin by describing the characteristic cycle of local cohomology modules supported on a face ideal. Then we give the main result, Theorem 3.8, which is a closed formula for the characteristic cycle of any local cohomology module supported on a monomial ideal in terms of the characteristic cycles of the local cohomology modules supported on sums of the face ideals appearing in the minimal primary decomposition.

As a consequence, we can decide when a given local cohomology module vanishes and compute the local cohomological dimension $cd(R,I)$ by means of these sums of face ideals, see Corollaries 3.12 and 3.13. We may also give a criterion for the Cohen–Macaulayness of R/I in terms of the minimal primary decompositon, see Corollary 3.11.

Section 4 is dedicated to compute the invariants $\lambda_{p,i}(R/I)$, that we shall call Lyubeznik numbers. Using [12] we prove that these numbers are exactly the multiplicities of the characteristic cycle of $H_{\mathfrak{m}}^p(H_I^{n-i}(R))$. Then we give a closed formula for these multiplicities, see Theorem 4.4, again in terms of the face ideals appearing in the minimal primary decomposition of I.

Finally, we give some examples and study combinatorial aspects of the Stanley–Reisner ring. In particular, we give a relation between the multiplicities of the characteristic cycle of the local cohomology modules supported on a squarefree monomial ideal and the f-vector of the associated simplicial complex. We remark with an example that these multiplicities are finer invariants than the f-vector.

We should mention that Barkats [1] gives an algorithm to compute a presentation of a local cohomology module supported on a monomial ideal. This algorithm has successfully been performed for ideals in $R=k[x_1,\dots,x_6]$. Then, Barkats can describe the characteristic cycle of these modules. Some results on Lyubeznik numbers have also been done by Walther for any ideal I. In [16] he describes all possible values of these numbers in the case $dim(R/I)\leq 2$. The same author, in [17], by using the theory of Gröbner basis on $D(R,k)$-modules, gives algorithms to determine the structure of $H_I^i(R)$ and $H_{\mathfrak{m}}^p(H_I^i(R))$ for arbitrary i, p and find $\lambda_{p,i}(R/I)$. Finally, Garcia and Sabbah [6], express the Lyubeznik numbers of the local ring of a complex isolated singularity in terms of Betti numbers of the associated real link.

2. $\mathscr{D}$-modules

In this section we fix some notation and recall several results on $\mathscr{D}$-modules. For details see [2, 3, 5].

Definition 2.1. Let k be a subring of a commutative noetherian ring R. The ring of differential operators, $D(R,k)$, is the subring of $Hom_k(R,R)$ generated by the k-linear derivations and the multiplications by elements of R.

By a $D(R,k)$-module we mean a left $D(R,k)$-module.

Set $R=k[x_1,\dots,x_n]$ or $R=k[[x_1,\dots,x_n]]$, where k is a field of characteristic zero and $x_1,\dots,x_n$ are indepentent variables. In both cases $D(R,k)$ is the non commutative R-algebra generated by the partial derivatives $\partial_i=\mathrm{d}/\mathrm{d}x_i$, with the relations given by

(i) $\partial_i\partial j=\partial j\partial_i$.

(ii) $\partial_i r - r\partial_i=\mathrm{d}r/\mathrm{d}x_i$, where $r\in R$.

In the case $R=k[x_1,\dots,x_n]$, $D(R,k)$ is called the nth Weyl algebra and denoted $A_n(k)$. For simplicity we shall denote in both cases $\mathscr{D}=D(R,k)$. The left and right Noetherian ring $\mathscr{D}$ has an increasing filtration $\{\Sigma_v\}_{v\geq 0}$ such that the corresponding associated graded ring $gr_\Sigma(\mathscr{D})=\Sigma_0\oplus\Sigma_1/\Sigma_0\oplus\cdots$ is isomorphic to the polynomial ring $R[y_1,\dots,y_n]$ where $y_i=\bar{\partial}_i\in\Sigma_1/\Sigma_0$.

A finitely generated $\mathscr{D}$-module M has a good filtration, i.e. M has an increasing sequence of finitely generated R-submodules $\Gamma_1\subseteq\Gamma_2\subseteq\cdots\subseteq M$ satisfying $\forall k,\ v\geq 0$:

(i) $\bigcup\Gamma_k=M$,

(ii) $\sum_v\Gamma_k\subseteq\Gamma_{v+k}$,

such that $gr_\Gamma(M)$ is a finitely generated $gr_\Sigma(\mathscr{D})$-module. The dimension of $gr_\Gamma(M)$ is independent of the good filtration on M. This integer is denoted by $d(M)$ and called the dimension of M. Bernstein's inequality states that $d(M)\geq n$ for every non-zero finitely generated $\mathscr{D}$-module M.

Definition 2.2. Let M be a finitely generated $\mathscr{D}$-module. One says that M is holonomic if $M=0$ or $d(M)=n$.

The class of holonomic modules has many good properties that we shall use in this paper. We list them in the following:

Theorem 2.3. (i) *Holonomic modules form a full abelian subcategory of the category of $\mathscr{D}$-modules. In particular, if*

$$0\to M_1\to M_2\to M_3\to 0$$

is an exact sequence of $\mathscr{D}$-modules, then M_2 is holonomic if and only if M_1 and M_3 are both holonomic.

(ii) *If M is holonomic then M has finite length as a $\mathscr{D}$-module.*

(iii) *R with its natural structure of $\mathscr{D}$-module is holonomic.*

(iv) *If M is holonomic and $f\in R$, then M_f is holonomic.*

(v) *If M is holonomic and $I=(f_1,\dots,f_r)\subset R$ then, from the Čech complex*

$$0\to M\to \oplus M_{f_i}\to \oplus M_{f_if_j}\to\cdots$$

one concludes by (i) and (iv) that $H_I^i(M)$ is holonomic for all i.

Remark 2.4. The ring R, all the localizations of R by any element of R, as well as the local cohomology modules $H_I^r(R)$ are, in fact, regular holonomic modules in the sense of Mebkhout [13].

The injective hull of the residue field k, $E_R(k)$ is regular holonomic since, in this case, it is isomorphic to $H_{\mathfrak{m}}^n(R)$. Lyubeznik [12, Proposition 2.3] obtains a presentation of $E_R(k)$ as $\mathscr{D}$-module in the following case:

Proposition 2.5. *Let $R=k[[x_1,\dots,x_n]]$ and $\mathfrak{m}$ be the maximal ideal of R. Then as an R-module, $\mathscr{D}/\mathscr{D}\mathfrak{m}$ is isomorphic to $E_R(k)$, the injective hull of the residue field of R in the category of R-modules.*

2.1. Characteristic cycle

Throughout the rest of this section we shall consider the ring $R=\mathbb{C}[x_1,\dots,x_n]$ of polynomials over the field of complex numbers. In this case $gr_\Sigma(\mathscr{D})=\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]$. Let M be a finitely generated $\mathscr{D}$-module and Γ a good filtration on M. Consider the ideal in $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]$:

$$J(M)=\sqrt{Ann_{gr_\Sigma(\mathscr{D})}(gr_\Gamma(M))}.$$

$J(M)$ is said to be the characteristic ideal of M. One can prove that $J(M)$ is independent of the good filtration on M.

Definition 2.6. The characteristic variety of M is the closed algebraic set:

$$C(M)=V(J(M))\subseteq\ Spec(gr_\Sigma(\mathscr{D})).$$

Remark 2.7. Let $X = Spec(R) = \mathbb{C}^n$. The characteristic variety $C(M)$ is a conical Lagrangian subvariety of $T^*X = Spec\,(gr_\Sigma(\mathscr{D})) = \mathbb{C}^{2n}$, see [10] for details. The study of the characteristic variety $C(M)$ allows us to get the support of the module M because if we consider the projection $\pi : T^*X \to X$ then $\pi(C(M)) = \mathrm{Supp}_R\, M$.

Definition 2.8. The characteristic cycle of M is defined as

$$CC(M) = \sum m_i V_i$$

where the sum is taken over all the irreducible components V_i of $C(M)$ and the m_i's are the corresponding multiplicities.

Remark 2.9. By the work of Pham [14] we can describe every irreducible component V_i of $C(M)$ as the conormal bundle $T^*_{Y_i}X$ relative to an irreducible subvariety Y_i of $X = Spec(R)$. Thus we can write

$$CC(M) = \sum m_i T^*_{Y_i}X.$$

Recall that $\pi(T^*_{Y_i}X) = Y_i$.

Proposition 2.10. *The characteristic cycle has the following properties*:

(i) *If* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is an exact sequence of holonomic* $\mathscr{D}$*-modules, then* $CC(M_2) = CC(M_1) + CC(M_3)$.

(ii) $CC(M) = 0$ *if and only if* $M = 0$.

2.1.1. The characteristic cycle of R

Consider the following filtration of R:

$$\Gamma^0 = \Gamma^1 = \cdots = R.$$

We have

$$gr_\Gamma(R) = R = R[y_1, \ldots, y_n]/(y_1, \ldots, y_n),$$

so Γ is a good filtration, and

$$J(M) = Ann_{gr_\Sigma(\mathscr{D})}(gr_\Gamma(R)) = (y_1, \ldots, y_n).$$

It is easy to see that $C(R) = T^*_X X$, so $CC(R) = T^*_X X$.

2.1.2. The characteristic cycle of R_f

Starting from a result of Ginsburg [7, Theorem 3.3], Briançon et al. [4] give a geometric formula of the characteristic cycle of M_f in terms of the characteristc cycle of M, where M is a regular holonomic $\mathscr{D}$-module and $f \in R$.

Theorem 2.11 (Briançon et al. [4]). *Let* $CC(M) = \sum m_i T^*_{Y_i}X$ *be the characteristic cycle of a regular holonomic* $\mathscr{D}$*-module M. Considering the divisor defined by f on the* *conormal bundle relative to* f $T_{f|_{Y_i}}$ *and the irreducible components,* $\Gamma_{i,j}$*, of this divisor with* $m_{i,j}$ *the multiplicity of the ideal defined by* $\pi(\Gamma_{i,j})$*, let*

$$\Gamma_i = \sum m_{i,j}\Gamma_{i,j}.$$

Then the characteristic cycle of M_f *is*

$$CC(M_f) = \sum_{f(Y_i)\neq 0} m_i(\Gamma_i + T^*_{Y_i}X).$$

We are going to apply this theorem when f is a monomial on the variables $x_1,\dots,x_n$. We first consider the following two cases:

Case 1: $M=R$ and $f=x_1$. We have $CC(R)=T^*_XX$, so in this case $Y=X$ and $f(X)\neq 0$. By definition, Y^0 is the non-singular part of X where f is a submersion. We must look for the points in X such that the gradient of f is different from zero. Since $\nabla f=(1,0,\dots,0)$ and X is non-singular, we have $Y^0=X$. Denote by $\mathscr{C}$ the hypersurface defined by $(f)^{-1}(f(x_1,x_2,\dots,x_n))$. Then $T_{f|_X}\subseteq T^*X$ is the closure of

$$\{v\in T^*X;\ z=\pi(v)\in X \text{ and } v \text{ annihilates } T_z\mathscr{C}\}.$$

Since $T_z\mathscr{C}=\langle(0,-1,\dots,0),\dots,(0,0,\dots,-1)\rangle$ we have

$$T_{f|_X}=\{(x_1,x_2,\dots,x_n,\alpha_1,\alpha_2,\dots,\alpha_n)\in T^*X;\ \alpha_2=\cdots=\alpha_n=0\}.$$

So the divisor defined by f on $T_{f|_X}$ is

$$\Gamma=\{(x_1,x_2,\dots,x_n,\alpha_1,\alpha_2,\dots,\alpha_n)\in T^*X;\ \alpha_2=\cdots=\alpha_n=0,\ x_1=0\}.$$

Note that it has only a component with multiplicity 1, and it is easy to prove that $\Gamma=T^*_{\{x_1=0\}}X$. So

$$CC(R_{x_1})=T^*_XX+T^*_{\{x_1=0\}}X.$$

Case 2: M is a regular holonomic $\mathscr{D}$-module such that $CC(M)=T^*_{\{x_1=0\}}X$ and $f=x_2$. In this case $Y=Y^0=\{x_1=0\}$ and $f(Y)\neq 0$. Now $\mathscr{C}$ is the hypersurface defined by $(f|_{\{x_1=0\}})^{-1}(f(x_1,x_2,\dots,x_n))$, so

$$T_{f|_Y}=\{(x_1,x_2,\dots,x_n,\alpha_1,\alpha_2,\dots,\alpha_n)\in T^*X;\ \alpha_3=\cdots=\alpha_n=0,\ x_1=0\}.$$

The divisor defined by f on $T_{f|_Y}$ is

$$\Gamma=\{(x_1,x_2,\dots,x_n,\alpha_1,\alpha_2,\dots,\alpha_n)\in T^*X;$$
$$\alpha_3=\cdots=\alpha_n=0,\ x_1=x_2=0\}=T^*_{\{x_1=x_2=0\}}X.$$

So $CC(M_{x_2})=T^*_{\{x_1=0\}}X+T^*_{\{x_1=x_2=0\}}X$.

By using this two cases we can find the characteristic cycle of R_f for any monomial f. Let $R=\mathbb{C}[x_1,\dots,x_n]$ and $f=x_1\cdots x_s,\ s\leq n$. Then

$$\begin{aligned}CC(R_f)=T^*_XX&+T^*_{\{x_1=0\}}X+\cdots+T^*_{\{x_s=0\}}X+T^*_{\{x_1=x_2=0\}}X\\&+\cdots+T^*_{\{x_{s-1}=x_s=0\}}X+\cdots+T^*_{\{x_1=\cdots=x_s=0\}}X.\end{aligned}$$

3. The main result

Let $R=k[x_1,\dots,x_n]$ or $R=k[[x_1,\dots,x_n]]$, where k is a field of characteristic zero. In this section we want to describe the characteristic cycle of the local cohomology modules $H_I^i(R)$ for a given ideal $I\subseteq R$ generated by monomials.

3.1. Cohen–Macaulay case

We begin with the following result, that is a consequence of [11, Theorem 1] (see also [1, Theorem 5.4.2.2]). In Corollary 3.11 we will reformulate it in terms of the minimal primary decomposition of a monomial ideal.

Recall that $cd(R,I):=\sup\{i;\ H_I^i(R)\neq 0\}$ is the cohomological dimension of R with respect to I.

Proposition 3.1. *Let $R=k[x_1,\dots,x_n]$ or $R=k[[x_1,\dots,x_n]]$, where k is a field of characteristic zero. For any ideal $I\subseteq R$ generated by squarefree monomials, the following are equivalent*:

(i) *R/I is Cohen–Macaulay.*

(ii) *$H_I^r(R)=0$ for any $r\neq ht\, I$.*

Proof. By completion we only have to consider the case $R=k[[x_1,\dots,x_n]]$. We have $\inf\{i;\ H_I^i(R)\neq 0\}=grade\, I=ht\, I$. On the other hand, by [11, Theorem 1(iv)] we have $cd(R,I)=n-depth_R(R/I)=ht\, I$. $\square$

This result enables us to compute the characteristic cycle $CC(H_I^h(R))$, when I is a monomial ideal of height h such that R/I is Cohen–Macaulay.

Proposition 3.2. *Let $I\in R$ be an ideal generated by squarefree monomials $f_1,\dots,f_r$ and $h=ht\, I$. Consider the Čech complex*:

$$0\to R\xrightarrow{d_0}\bigoplus R_{f_i}\xrightarrow{d_1}\bigoplus R_{f_if_k}\xrightarrow{d_2}\cdots\xrightarrow{d_{r-1}}R_{f_1\cdots f_r}\to 0$$

and denote $R_j=\bigoplus R_{f_{i_1}\cdots f_{i_j}}$, for $j=0,\dots,r$. If R/I is Cohen–Macaulay, then

$$CC(H_I^h(R))=CC(R_h)-CC(R_{h+1})+\cdots+(-1)^{r-h}CC(R_r)$$
$$-CC(R_{h-1})+\cdots+(-1)^hCC(R_0).$$

Proof. By Proposition 3.1 $H_I^r(R)=0$ for any $r\neq h$. Then we have

$$0=H_I^0(R)=Ker\, d_0.$$

So $CC(\mathrm{Im}\, d_0)=CC(R_0)$. Similarly, $0=H_I^1(R)=Ker\, d_1/\mathrm{Im}\, d_0$ and so $CC(\mathrm{Im}\, d_1)=CC(R_1)-CC(R_0)$. By repeating this argument we obtain

$$CC(\mathrm{Im}\, d_{h-1})=CC(R_{h-1})-CC(R_{h-2})+\cdots+(-1)^hCC(R_0).$$

On the other hand, if $r > h$, we have

$$0 = H_I^r(R) = Ker\, d_r/\mathrm{Im}\, d_{r-1}, \text{ so } CC(Ker\, d_r) = CC(R_r).$$

As before

$$CC(Ker\, d_h) = CC(R_h) - CC(R_{h+1}) + \cdots + (-1)^{r-h} CC(R_r).$$

Since $H_I^h(R) = Ker\, d_h/\mathrm{Im}\, d_{h-1}$, we get the formula. $\square$

If I is a complete intersection we then have:

Corollary 3.3. *Assume I is a complete intersection. Then,*

$$CC(H_I^h(R)) = CC(R_h) - CC(R_{h-1}) + \cdots + (-1)^h CC(R_0).$$

$CC(R_i)$ have been computed in Section 2.1.2 when $R = \mathbb{C}[x_1,\ldots,x_n]$. Then we can use this corollary to compute the characteristic cycle of $H_I^h(R)$, where $I = (x_{i_1},\ldots,x_{i_h})$ is a face ideal. Namely, we have

$$CC(H_I^h(R)) = T^*_{\{x_{i_1} = \cdots = x_{i_h} = 0\}} X.$$

Remark 3.4. We can extend this result to the case $R = k[x_1,\ldots,x_n]$ or $R = k[[x_1,\ldots,x_n]]$, where k is a field of characteristic zero in the following way:

Case 1: For $L = \mathbb{Q}, \mathbb{C}, k$ denote $R_L = L[x_1,\ldots,x_n]$, and for a face ideal $I \subseteq R_L$ let $M_L = H_I^h(R_L)$. We can consider every face ideal as $I \subseteq R_{\mathbb{Q}}$ and consider the $A_n(\mathbb{Q})$-module $M_{\mathbb{Q}} = H_I^h(R_{\mathbb{Q}})$. By flat base change we have

$$M_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = H_I^h(R_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} = H_I^h(R_{\mathbb{C}}) = M_{\mathbb{C}},$$
$$M_{\mathbb{Q}} \otimes_{\mathbb{Q}} k = H_I^h(R_{\mathbb{Q}}) \otimes_{\mathbb{Q}} k = H_I^h(R_k) = M_k.$$

So we get the characteristic ideals $J(M_{\mathbb{C}})$ and $J(M_k)$ as extensions of the characteristic ideal $J(M_{\mathbb{Q}})$, respectively, to $\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ and $k[x_1,\ldots,x_n,y_1,\ldots,y_n]$. Then,

$$CC(H_I^h(R_k)) = T^*_{\{x_{i_1} = \cdots = x_{i_h} = 0\}} X,$$

where $X = Spec\, k[x_1,\ldots,x_n]$.

Case 2: Let $\hat{R} = k[[x_1,\ldots,x_n]]$. For a face ideal $I \subseteq R = k[x_1,\ldots,x_n]$ consider a good filtration $\{\Gamma_v\}$ on $M = H_I^h(R)$. Then $\{\hat{\Gamma}_v\} = \{\Gamma_v \otimes_R \hat{R}\}$ is a good filtration on $\hat{M} = M \otimes_R \hat{R}$, and $gr_{\hat{\Gamma}} \hat{M} = gr_{\Gamma} M \otimes_R \hat{R}$.

By flat base change $\hat{M} = H_I^h(\hat{R})$ and so extending the characteristic ideal of M to $k[[x_1,\ldots,x_n]][y_1,\ldots,y_n]$ we see that

$$CC(H_I^h(\hat{R})) = T^*_{\{x_{i_1} = \cdots = x_{i_h} = 0\}} X,$$

where $X = Spec\, k[[x_1,\ldots,x_n]]$.

3.2. General case

3.2.1. Sum of face ideals

Let $I \subseteq R$ be an ideal generated by squarefree monomials. Consider the minimal primary decompositon $I = I_1 \cap \cdots \cap I_m$.

The ideals I_i are face ideals of the form $(x_{i_1}, \dots, x_{i_h})$. These are complete intersection ideals of height h. The sum of face ideals $I_{i_1} + \cdots + I_{i_s}$ is again a face ideal, and $ht(I_{i_1} + \cdots + I_{i_s}) \leq ht\, I_{i_1} + \cdots + ht\, I_{i_s}$. From now on we shall denote $h_{i_1 \cdots i_s} := ht(I_{i_1} + \cdots + I_{i_s})$.

We are going to describe some sets of sums of the face ideals in the minimal primary decomposition of I. They appear in a natural way when we make an iterated use of the Mayer–Vietoris sequence.

Let $I = I_1 \cap \cdots \cap I_m$. Then we define $\omega = \{\omega_1, \omega_2, \dots, \omega_m\}$, where

$$\begin{aligned} \omega_1 &= \{I_1, \dots, I_m\}, \\ \omega_2 &= \{I_1 + I_2, \dots, I_{m-1} + I_m\}, \\ &\vdots \\ \omega_m &= \{I_1 + I_2 + \cdots + I_m\}. \end{aligned}$$

Similarly, we define the set of sums of face ideals with I_m as a summand, $\omega^m = \{\omega_1^m, \omega_2^m, \dots, \omega_m^m\}$, where

$$\begin{aligned} \omega_1^m &= \{I_m\}, \\ \omega_2^m &= \{I_1 + I_m, \dots, I_{m-1} + I_m\}, \\ &\vdots \\ \omega_m^m &= \{I_1 + I_2 + \cdots + I_m\}. \end{aligned}$$

Definition 3.5. We say that $I_{i_1} + \cdots + I_{i_j} \in \omega_j$ and $I_{i_1} + \cdots + I_{i_j} + I_{i_{j+1}} \in \omega_{j+1}$ are paired if $h_{i_1 \dots i_j} = h_{i_1 \dots i_{j+1}}$. Equivalently, if $I_{i_1} + \cdots + I_{i_j} = I_{i_1} + \cdots + I_{i_j} + I_{i_{j+1}}$, i.e., a generator of $I_{i_{j+1}}$ is also a generator of I_{i_k} for some $k = 1, \dots, j$.

Definition 3.6. We say that $I_{i_1} + \cdots + I_{i_j} \in \omega_j$ and $I_{i_1} + \cdots + I_{i_j} + I_{i_{j+1}} \in \omega_{j+1}$ are almost paired if $h_{i_1 \dots i_j} + 1 = h_{i_1 \dots i_{j+1}}$. Equivalently, if there is a generator of $I_{i_{j+1}}$ which is not a generator of I_{i_k} for all $k = 1, \dots, j$.

The formula we shall obtain in Theorem 3.8 for the characteristic cycle of a local cohomology module supported on a monomial ideal will be given in terms of non-paired sums of the face ideals in the minimal primary decomposition. The following algorithm allows to obtain these sums.

Algorithm 1

Input: $\omega =$ set of sums of face ideals

Output: $\Omega =$ set of non paired sums of face ideals

When we compare a set of sums of face ideals S with a set of sums of face ideals S' we mean the following: If a sum in S is paired with a sum in S' we omit this pair of sums, so repeating this process no sum of face ideals of S is paired to a sum of face ideals of S' at the end. The algorithm is then as follows:

Step 1: (1.1) Compare I_1+I_2 with $\{I_1,I_2\}$. Save the result in S_0^1.

(1.2) For $k=1,\dots,m-2$; $i_1=3,\dots,m-(k-1)$; $i_2=i_1+1,\dots,m-(k-2);\dots$; $i_k=i_{k-1}+1,\dots,m$; compare $I_1+I_2+I_{i_1}+\cdots+I_{i_k}$ with $\{I_1+I_{i_1}+\cdots+I_{i_k}, I_2+I_{i_1}+\cdots+I_{i_k}\}$. Save the result in $S^1_{i_1\cdots i_k}$.

For $j=3,\dots,m$;

Step $j-1$. ($j-1.1$) Compare S_j^{j-2} with $\{S_0^{j-2}, I_j\}$. Save the result in S_0^{j-1}.

($j-1.2$) For $k=1,\dots,m-j$; $i_1=j+1,\dots,m-(k-1)$; $i_2=i_1+1,\dots,m-(k-2);\dots$; $i_k=i_{k-1}+1,\dots,m$; compare $S^{j-2}_{ji_1\dots i_k}$ with $\{S^{j-2}_{i_1\dots i_k}, I_j+I_{i_1}+\cdots+I_{i_k}\}$. Save the result in $S^{j-1}_{i_1\dots i_k}$.

The last set of non-paired ideals that we obtain by using this algorithm is S_0^{m-1}. Collecting the sums of face ideals of S_0^{m-1} with j summands in Ω_j we get $\Omega=\{\Omega_1,\Omega_2,\dots,\Omega_m\}$, in such a way that no sum in Ω_j is paired with a sum in Ω_{j+1}.

We can use a similar algorithm to get the sets of non-paired sums of face ideals with the face ideal I_m as a summand, $\Omega^m=\{\Omega_1^m,\Omega_2^m,\dots,\Omega_m^m\}$.

Finally, we define the sets of non-paired sums of face ideals with a given height $\Omega_{j,r}:=\{I_{i_1}+\cdots+I_{i_j}\in\Omega_j\colon h_{i_1\cdots i_j}=r+(j-1)\}$. The formula we shall give in Theorem 3.8 will be expressed in terms of these sets of non-paired sums of face ideals.

Remark 3.7. One can see that the sets $\Omega_{j,r}$ are independent of the sums of face ideals chosen to compare in each step.

We can use a similar algorithm to get the set of non paired sums of face ideals with the face ideal I_m as a summand $\Omega^m=\{\Omega_1^m,\Omega_2^m,\dots,\Omega_m^m\}$, and with a given height $\Omega^m_{j,r}$ $j=1,\dots,m$.

Now we are ready to formulate our main result.

Theorem 3.8. *Let $I\subseteq R$ be an ideal generated by squarefree monomials and let $I=I_1\cap\cdots\cap I_m$ be the minimal primary decompositon. Then,*

$$CC(H_I^r(R))=\sum_{I_i\in\Omega_{1,r}} CC(H^r_{I_i}(R))+\sum_{I_i+I_j\in\Omega_{2,r}} CC(H^{r+1}_{I_i+I_j}(R))+\cdots$$
$$+\sum_{I_1+\cdots+I_m\in\Omega_{m,r}} CC(H^{r+(m-1)}_{I_1+\cdots+I_m}(R)).$$

Proof. We shall proceed by induction on m, the number of ideals in the minimal primary decomposition, being the case $m=1$ trivial. To do it we shall split a Mayer–Vietoris sequence of the type

$$\cdots\to H^r_{U+V}(R)\to H^r_U(R)\oplus H^r_V(R)\to H^r_{U\cap V}(R)\to H^{r+1}_{U+V}(R)\to\cdots$$

into short exact sequences of kernels and cokernels:

$$0 \to B_r \to H^r_U(R) \oplus H^r_V(R) \to C_r \to 0,$$

$$0 \to C_r \to H^r_{U\cap V}(R) \to A_{r+1} \to 0,$$

$$0 \to A_{r+1} \to H^{r+1}_{U+V}(R) \to B_{r+1} \to 0,$$

so,

$$CC(H^r_{U\cap V}(R)) = CC(C_r) + CC(A_{r+1}).$$

Assume we have proved the formula for ideals with less terms than m in the minimal primary decomposition. Now consider

$$U = I_1 \cap \cdots \cap I_{m-1},$$

$$V = I_m,$$

$$U \cap V = I = I_1 \cap \cdots \cap I_m,$$

$$U + V = I_1 \cap \cdots \cap I_{m-1} + I_m.$$

By induction we have determined $CC(H^r_U(R))$ and $CC(H^r_V(R))$. Now we shall describe $CC(H^r_{U+V}(R))$.

Lemma 3.9. *Let $I \subseteq R$ be generated by squarefree monomials and $I = I_1 \cap \cdots \cap I_m$ be the minimal primary decompositon. Then*

$$CC(H^r_{I_1\cap\cdots\cap I_{m-1}+I_m}(R)) = \sum_{I_i+I_m\in\Omega^m_{2,r}} CC(H^r_{I_i+I_m}(R)) + \cdots + \sum_{I_1+\cdots+I_m\in\Omega^m_{m,r}} CC(H^{r+(m-2)}_{I_1+\cdots+I_m}(R)).$$

Remark 3.10. The formulas in Theorem 3.8 and Lemma 3.9 show that if $\alpha \in CC(H^r_I(R))$ then $\alpha \notin CC(H^j_I(R))$ for $j \neq r$.

Proof of Lemma 3.9. We use induction on the number of components in the minimal primary decompositon. We shall consider a Mayer–Vietoris sequence with

$$U = I_1 \cap \cdots \cap I_{m-2} + I_m,$$

$$V = I_{m-1} + I_m,$$

$$U \cap V = I_1 \cap \cdots \cap I_{m-1} + I_m,$$

$$U + V = I_1 \cap \cdots \cap I_{m-2} + (I_{m-1} + I_m).$$

We have by induction

$$CC(H^r_{I_1\cap\cdots\cap I_{m-2}+I_m}(R)) = \sum_{I_i+I_m\in\Omega^m_{2,r}} CC(H^r_{I_i+I_m}(R)) + \cdots + \sum_{I_1+\cdots+I_m\in\Omega^m_{m,r}} CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-2}+I_m}(R)),$$

$$CC(H^r_{I_{m-1}+I_m}(R)) = CC(H^r_{I_{m-1}+I_m}(R))$$

and

$$CC(H^r_{I_1\cap\cdots\cap I_{m-2}+(I_{m-1}+I_m)}(R)) = \sum_{I_i+I_{m-1}+I_m\in\Omega^{(m-1)+m}_{2,r}} CC(H^r_{I_i+I_{m-1}+I_m}(R)) + \cdots + \sum_{I_1+\cdots+I_m\in\Omega^{(m-1)+m}_{m,r}} CC(H^{r+(m-3)}_{I_1+\cdots+I_m}(R)).$$

Note that we use the sets $\Omega^{(m-1)+m}_{j,r}$ of non-paired sums of face ideals with the face ideal $I_{m-1}+I_m$ as a summand to describe $CC(H^r_{U+V}(R))$. To get the desired formula we only need to describe $CC(B_r)$, so it is enough to prove the following:

Claim.

$$CC(B_r) = \sum T^*_{\{x_{i_1}=\cdots=x_{i_j}=0\}}X,$$

where the sum is taken over the cycles

$$T^*_{\{x_{i_1}=\cdots=x_{i_j}=0\}}X \in CC(H^r_U(R)\oplus H^r_V(R)) \cap CC(H^r_{U+V}(R)).$$

The inclusion $\subseteq$ is obvious. To prove the other one let $\alpha = T^*_{\{x_{i_1}=\cdots=x_{i_j}=0\}}X \in CC(H^r_U(R)\oplus H^r_V(R)) \cap CC(H^r_{U+V}(R))$ and suppose $\alpha\notin CC(B_r)$. We must consider the sum of face ideals in the minimal primary decomposition that we need to express α as the characteristic cycle of a local cohomology module supported on this sum.

Case 1. $\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_m}(R)) = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_{m-1}+I_m}(R))$, where $s<m-2$.

(1.1) If there exists an integer $l\in\{1,\dots,m-2\}\setminus\{i_1,\dots,i_s\}$ such that $\alpha \neq CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_l+I_m}(R))$, then I_l is not involved in expressing α and one can consider the following Mayer–Vietoris sequence:

$$U = I_1\cap\cdots\cap\hat{I}_l\cap\cdots\cap I_{m-2}+I_m,$$

$$V = I_{m-1}+I_m,$$

$$U\cap V = I_1\cap\cdots\cap\hat{I}_l\cap\cdots\cap I_{m-1}+I_m,$$

$$U+V = I_1\cap\cdots\cap\hat{I}_l\cap\cdots\cap I_{m-2}+(I_{m-1}+I_m).$$

Note that in this sequence $\alpha\in CC(H^r_U(R)\oplus H^r_V(R))\cap CC(H^r_{U+V}(R))$ and $\alpha\notin CC(B_r)$.

By induction and using Remark 3.10 there is a contradiction since

$$\alpha \in CC(H^r_{U+V}(R)) \text{ and } \alpha \notin CC(B_r) \Rightarrow \alpha \in CC(H^{r-1}_{U\cap V}(R)),$$

$$\alpha \in CC(H^r_U(R) \oplus H^r_V(R)) \text{ and } \alpha \notin B_r \Rightarrow \alpha \in CC(H^r_{U\cap V}(R)).$$

(1.2) If for any $l \in \{1,\dots,m-2\}\backslash\{i_1,\dots,i_s\}$ we have $\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_l+I_m}(R))$ then,

$$\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_l+I_m}(R)) \notin CC(H^r_{I_1\cap\cdots\cap I_{m-2}+I_m}(R) \oplus H^r_{I_{m-1}+I_m}(R))$$

and so there is a previous induction step in computing

$$CC(H^r_{I_1\cap\cdots\cap I_{m-2}+I_m}(R) \oplus H^r_{I_{m-1}+I_m}(R)),$$

such that $\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_l+I_m}(R)) \in CC(B_r)$. Recall that $CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_m}(R)) \notin CC(B_r)$ for any step, so there exists J, sum of face ideals with $I_l + I_m$ as a summand, such that

$$\alpha = CC(H^{r+(s-1)}_J(R)) \notin CC(H^r_{I_1\cap\cdots\cap I_{m-2}+I_m}(R) \oplus H^r_{I_{m-1}+I_m}(R)).$$

We also have

$$\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_l+I_{m-1}+I_m}(R)) \notin CC(H^r_{I_1\cap\cdots\cap I_{m-2}+(I_{m-1}+I_m)}(R)),$$

so $CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_l+I_{m-1}+I_m}(R)) \in CC(B_r)$ in the corresponding induction step and

$$\alpha = CC(H^{r+(s-1)}_{J+I_{m-1}}(R)) \notin CC(H^r_{I_1\cap\cdots\cap I_{m-2}+(I_{m-1}+I_m)}(R)).$$

Then, when we use the Mayer–Vietoris sequence with

$$U = I_1 \cap \cdots \cap \hat{I}_l \cap \cdots \cap I_{m-2} + I_m,$$

$$V = I_{m-1} + I_m,$$

$$U \cap V = I_1 \cap \cdots \cap \hat{I}_l \cap \cdots \cap I_{m-1} + I_m,$$

$$U + V = I_1 \cap \cdots \cap \hat{I}_l \cap \cdots \cap I_{m-2} + (I_{m-1} + I_m),$$

we have

$$\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_m}(R)) \in CC(H^r_U(R) \oplus H^r_V(R)),$$

$$\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_{m-1}+I_m}(R)) \in CC(H^r_{U+V}(R))$$

and $\alpha \notin CC(B_r)$, and so we get a contradiction.

Case 2: $\alpha = CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-2}+I_m}(R)) = CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-2}+I_{m-1}+I_m}(R))$.

In this case we consider the Mayer–Vietoris sequence with

$$U = (I_1 + \cdots + I_{m-2}) + I_m,$$

$$V = I_{m-1} + I_m,$$

$$U \cap V = (I_1 + \cdots + I_{m-2}) \cap I_{m-1} + I_m,$$

$$U + V = (I_1 + \cdots + I_{m-2}) + (I_{m-1} + I_m).$$

we have

$$\alpha = CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-2}+I_m}(R)) \in CC(H^r_U(R) \oplus H^r_V(R)),$$

$$\alpha = CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-2}+I_{m-1}+I_m}(R)) \in CC(H^r_{U+V}(R))$$

and $\alpha \notin CC(B_r)$, so we get a contradiction and this proves the claim. $\square$

Now, we may continue the proof of Theorem 3.8. Recall that we use induction on the number of components in the minimal primary decompositon and the Mayer–Vietoris sequence:

$$\cdots \to H^r_{I_1\cap\cdots\cap I_{m-1}}(R) \oplus H^r_{I_m}(R) \to H^r_I(R) \to H^{r+1}_{I_1\cap\cdots\cap I_{m-1}+I_m}(R) \to \cdots .$$

By Lemma 3.9 and induction we have

$$CC(H^r_{I_1\cap\cdots\cap I_{m-1}}(R)) = \sum_{I_i\in\Omega_{1,r}} CC(H^r_{I_i}(R)) + \cdots + \sum_{I_1+\cdots+I_{m-1}\in\Omega_{m-1,r}} CC(H^{r+(m-2)}_{I_1+\cdots+I_{m-1}}(R)),$$

$$CC(H^r_{I_1\cap\cdots\cap I_{m-1}+I_m}(R)) = \sum_{I_i+I_m\in\Omega^m_{2,r}} CC(H^r_{I_i+I_m}(R)) + \cdots + \sum_{I_1+\cdots+I_m\in\Omega^m_{m,r}} CC(H^{r+(m-2)}_{I_1+\cdots+I_m}(R)).$$

To finish the proof of Theorem 3.8 we must only describe $CC(B_r)$.

Claim.

$$CC(B_r) = \sum T^*_{\{x_{i_1}=\cdots=x_{i_j}=0\}}X,$$

where the sum is taken over the cycles

$$T^*_{\{x_{i_1}=\cdots=x_{i_j}=0\}}X \in CC(H^r_{I_1\cap\cdots\cap I_{m-1}}(R) \oplus H^r_{I_{m-1}+I_m}(R)) \cap CC(H^r_{I_1\cap\cdots\cap I_{m-1}+I_m}(R)).$$

The proof of this claim is as in Lemma 3.9. $\square$

By using Theorem 3.8 we can also reformulate Proposition 3.1 in terms of the sets of non-paired sums of face ideals.

Corollary 3.11. *Let $R = k[x_1, \dots, x_n]$ or $R = k[[x_1, \dots, x_n]]$, where k is a field of characteristic zero. For an ideal $I \subseteq R$ generated by squarefree monomials, the following are equivalent:*
(i) *R/I is Cohen–Macaulay.*
(ii) *$H_I^r(R) = 0$ for any $r \neq ht\, I$.*
(iii) *$\Omega_{j,r} = \emptyset$, for any $r \neq ht\, I$, $\forall j$.*
(iv) *For all $I_{i_1} + \cdots + I_{i_j} \in \Omega_j$, $h_{i_1 \cdots i_j} = ht\, I + (j-1)$.*

Note that Theorem 3.8 also provides a criterion to decide when $H_I^r(R)$ vanishes. By [11, Theorem 1(iii)] this is equivalent to determine when $Ext_R^r(R/I, R)$ vanishes.

Corollary 3.12. *Let $R = k[x_1, \dots, x_n]$ or $R = k[[x_1, \dots, x_n]]$, where k is a field of characteristic zero. For an ideal $I \subseteq R$ generated by squarefree monomials, the following are equivalent:*
(i) *$H_I^r(R) \neq 0$.*
(ii) *There exists j such that $\Omega_{j,r} \neq \emptyset$.*
(iii) *There exists $I_{i_1} + \cdots + I_{i_j} \in \Omega_j$ such that $h_{i_1 \dots i_j} = r + (j-1)$.*

Corollary 3.13. *The cohomological dimension of R with respect to I is*

$$cd(R, I) = \max\{h_{i_1 \dots i_j} - (j-1) \colon I_{i_1} + \cdots + I_{i_j} \in \Omega_j\}.$$

Remark 3.14. By [11, Theorem 1(iv)], $cd(R, I) = proj.dim_R(R/I) = n - depth_R(R/I)$.

4. Lyubeznik numbers

Let A be a quotient of dimension d of the regular local ring $R = k[[x_1, \dots, x_n]]$, with $x_1, \dots, x_n$ independent variables. In this section we want to study the Lyubeznik numbers $\lambda_{p,i}(A)$ introduced in [12, Section 4]. In his paper Lyubeznik also gives some properties of these numbers:
(i) $\lambda_{p,i}(A) = 0$ if $i > d$.
(ii) $\lambda_{p,i}(A) = 0$ if $p > i$.
(iii) $\lambda_{d,d}(A) \neq 0$.
Walther [16] defines the type of R/I as the tringular matrix given by $\lambda_{p,i}(A)$.

$$\Lambda(A) = \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{pmatrix}.$$

Remark 4.1. By [12, Lemma 1.4]

$$\lambda_{p,i}(R/I) = \mu_p(\mathfrak{m}, H_I^{n-i}(R)) = \mu_0(\mathfrak{m}, H_{\mathfrak{m}}^p(H_I^{n-i}(R))).$$

4.1. Lyubeznik numbers and characteristic cycles

Let $R=k[[x_1,\dots,x_n]]$ and $I\subseteq R$ an ideal generated by monomials. In this section we shall compute the characteristic cycle of $H^p_{\mathfrak{m}}(H^r_I(R))$. Consider the ring of differential operators $\mathscr{D}=D(R,k)$. By [12, Theorem 3.4(a)]

$$H^p_{\mathfrak{m}}(H^{n-i}_I(R))=(\mathscr{D}/\mathscr{D}\mathfrak{m})^{\lambda_{p,i}(R/I)}=E_R(R/\mathfrak{m})^{\lambda_{p,i}(R/I)}.$$

Consider the filtration $\{\Sigma_v\}$ on $\mathscr{D}$. Then $\{\Sigma_v/\Sigma_v\cap\mathfrak{m}\}$ and $\{\Sigma_v\cap\mathfrak{m}\}$ are good filtrations on $\mathscr{D}/\mathscr{D}\mathfrak{m}$ and $\mathscr{D}\mathfrak{m}$, respectively. We have an exact sequence

$$0\to gr\,\mathscr{D}\mathfrak{m}\to gr\,\mathscr{D}\to gr\,\mathscr{D}/\mathscr{D}\mathfrak{m}\to 0.$$

Thus $gr\,\mathscr{D}/\mathscr{D}\mathfrak{m}\cong gr\,\mathscr{D}/gr\,\mathscr{D}\mathfrak{m}$, and one can see that the characteristic ideal is $J(\mathscr{D}/\mathscr{D}\mathfrak{m})=\mathfrak{m}$. So $CC(\mathscr{D}/\mathscr{D}\mathfrak{m})=T^*_{\{x_1=\cdots=x_n=0\}}X$.

Then $CC(H^p_{\mathfrak{m}}(H^{n-i}_I(R)))=\lambda_{p,i}T^*_{\{x_1=\cdots=x_n=0\}}X$.

Remark 4.2. If $I\subseteq R$ is a monomial ideal of height h such that R/I is Cohen–Macaulay then the spectral sequence $E_2^{p,q}=H^p_{\mathfrak{m}}(H^q_I(R))\Rightarrow H^{p+q}_{\mathfrak{m}}(R)$ gives

$$H^p_{\mathfrak{m}}(H^h_I(R))=\begin{cases}0 & \text{if } p\neq n-h,\\ E_R(R/\mathfrak{m}) & \text{if } p=n-h.\end{cases}$$

So the type is

$$\Lambda(R/I)=\begin{pmatrix}0 & \cdots & 0\\ & \ddots & \vdots\\ & & 1\end{pmatrix}.$$

Later we shall see with an example that the converse does not hold.

4.1.1. Sum ideals

The formula we shall obtain in Theorem 4.4 for the characteristic cycle of $H^p_{\mathfrak{m}}(H^r_I(R))$ will be given in terms of non-paired and non-almost paired sums of the face ideals in the minimal primary decomposition. We give an analogous algorithm to the one of Section 3.2.1 to get these sums.

Algorithm 2

Input: $\Omega=$ set of non-paired sums of face ideals.

Output: $\Gamma=$ set of non-paired and non-almost paired sums of face ideals.

The algorithm is as the one of Section 3.2.1 but now, when we compare a set of non-paired sums of face ideals S with a set of non-paired sums of face ideals S' we mean the following: If a sum in S is almost paired with a sum in S' we omit this pair of sums, so repeating this process no sum of non-paired face ideals of S is almost paired to a sum of non-paired face ideals of S' at the end.

Collecting by the number of summands the last set of non-paired and non-almost paired sums of face ideals obtained by using this algorithm we get $\Gamma=\{\Gamma_1,\Gamma_2,\dots,\Gamma_m\}$, in such a way that no sum in Γ_j is almost paired with a sum in Γ_{j+1}.

We also define the set of non-paired and non-almost paired sums of face ideals with a given height $\Gamma_{j,r}=\{I_{i_1}+\cdots+I_{i_j}\in\Gamma_j\colon\ h_{i_1\cdots i_j}=r+(j-1)\}$. We shall denote $h_{j,r}=r+(j-1)$. The formula we shall give in Theorem 4.4 will be expressed in terms of these sets of non-paired and non-almost paired sums of face ideals.

Remark 4.3. One can see that the numbers of face ideals in the sets $\Gamma_{j,r}$ are independent of the sums of face ideals chosen to compare in each step.

We can use a similar algorithm to get the sets of non-paired and non-almost paired sums of face ideals with the face ideal I_m as a summand, $\Gamma^m=\{\Gamma_1^m,\Gamma_2^m,\dots,\Gamma_m^m\}$, and with a given height $\Gamma_{j,r}^m$ for $j=1,\dots,m$.

Theorem 4.4. *Let $I\subseteq R=k[[x_1,\dots,x_n]]$ be an ideal generated by squarefree monomials and $I=I_1\cap\cdots\cap I_m$ be its minimal primary decompositon. Then*

$$CC(H_{\mathfrak{m}}^p(H_I^r(R)))=\lambda_{p,n-r}T^*_{\{x_1=\cdots=x_n=0\}}X,$$

where $\lambda_{p,n-r}=\sharp\Gamma_{j,r}$ such that $h_{j,r}=n-p$.

Proof. We are going to use similar ideas as in the proof of Theorem 3.8 and we shall use the same notation. Consider the Mayer–Vietoris sequence with

$$U=I_1\cap\cdots\cap I_{m-1},$$

$$V=I_m,$$

$$U\cap V=I=I_1\cap\cdots\cap I_m,$$

$$U+V=I_1\cap\cdots\cap I_{m-1}+I_m.$$

Then we have

$$0\to C_r\to H_{U\cap V}^r(R)\to A_{r+1}\to 0.$$

Applying the long exact sequence of local cohomology we get

$$\cdots\to H_{\mathfrak{m}}^p(C_r)\to H_{\mathfrak{m}}^p(H_{U\cap V}^r(R))\to H_{\mathfrak{m}}^p(A_{r+1})\to H_{\mathfrak{m}}^{p+1}(C_r)\to\cdots\ .$$

We shall split this sequence into short exact sequences of kernels and cokernels:

$$0\to Z_{p-1}\to H_{\mathfrak{m}}^p(C_r)\to X_p\to 0,$$

$$0\to X_p\to H_{\mathfrak{m}}^p(H_{U\cap V}^r(R))\to Y_p\to 0,$$

$$0\to Y_p\to H_{\mathfrak{m}}^p(A_{r+1})\to Z_p\to 0.$$

So $CC(H_{\mathfrak{m}}^p(H_{U\cap V}^r(R)))=CC(X_p)+CC(Y_p)$.

Assume we have proved the formula for ideals with less terms than m in the minimal primary decomposition. To compute $CC(H^p_{\mathfrak{m}}(A_{r+1}))$ we need to describe $CC(H^p_{\mathfrak{m}}(H^{r+1}_{I_1\cap\cdots\cap I_{m-1}+I_m}(R)))$.

Lemma 4.5. *Let $I \subseteq R = k[[x_1,\dots,x_n]]$ be generated by squarefree monomials and $I = I_1 \cap \cdots \cap I_m$ be the minimal primary decompositon. Then*

$$CC(H^p_{\mathfrak{m}}(H^r_{I_1\cap\cdots\cap I_{m-1}+I_m}(R))) = \lambda_{p,n-r} T^*_{\{x_1=\cdots=x_n=0\}}X,$$

where $\lambda_{p,n-r} = \sharp\Gamma^m_{j,r}$ such that $h^m_{j,r} = n-p$.

Proof. We shall use induction on m, the number of components in the minimal primary decompositon.

$m=2$. Let $h = ht(I_1+I_2)$. Since I_1+I_2 is a complete intersection, by using Remark 4.2 we obtain

$$CC(H^{n-h}_m(H^h_{I_1+I_2}(R))) = T^*_{\{x_1=\cdots=x_n=0\}}X.$$

$m>2$. Consider the Mayer–Vietoris sequence with

$$\begin{aligned} U &= I_1\cap\cdots\cap I_{m-2}+I_m,\\ V &= I_{m-1}+I_m,\\ U\cap V &= I_1\cap\cdots\cap I_{m-1}+I_m,\\ U+V &= I_1\cap\cdots\cap I_{m-2}+(I_{m-1}+I_m). \end{aligned}$$

Recall that $CC(C_r)\subseteq CC(H^r_U(R))+CC(H^r_V(R))$ and $CC(A_{r+1})\subseteq CC(H^{r+1}_{U+V}(R))$, where

$$CC(H^r_U(R)) = \sum_{I_i+I_m\in\Omega^m_{2,r}} CC(H^r_{I_i+I_m}(R)) + \cdots + \sum_{I_1+\cdots+I_m\in\Omega^m_{m,r}} CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-2}+I_m}(R)),$$

$$CC(H^r_V(R)) = CC(H^r_{I_{m-1}+I_m}(R))$$

and

$$\begin{aligned} CC(H^{r+1}_{U+V}(R)) = &\sum_{I_i+I_{m-1}+I_m\in\Omega^{(m-1)+m}_{2,r+1}} CC(H^{r+1}_{I_i+I_{m-1}+I_m}(R)) + \cdots\\ &+ \sum_{I_1+\cdots+I_m\in\Omega^{(m-1)+m}_{m,r+1}} CC(H^{r+(m-2)}_{I_1+\cdots+I_m}(R)). \end{aligned}$$

Note that we use the sets $\Omega^{(m-1)+m}_{j,r+1}$ of non-paired sums of face ideals with the face ideal $I_{m-1}+I_m$ as a summand to describe $CC(H^{r+1}_{U+V}(R))$. To get the desired formula we only need to describe $CC(Z_p)$, so it is enough to prove the following

Claim.

$$CC(Z_p) = \sum T^*_{\{x_1=\cdots=x_n=0\}}X,$$

where the sum is taken over the cycles

$$T^*_{\{x_1=\cdots=x_n=0\}}X = CC(H^p_{\mathfrak{m}}(H^{r+j}_{I_{i_1}+\cdots+I_{i_j}+I_{m-1}+I_m}(R))) \in CC(H^p_{\mathfrak{m}}(A_{r+1})),$$

such that

$$T^*_{\{x_1=\cdots=x_n=0\}}X = CC(H^{p+1}_{\mathfrak{m}}(H^{r+j-1}_{I_{i_1}+\cdots+I_{i_j}+I_m}(R))) \in CC(H^{p+1}_{\mathfrak{m}}(C_r)).$$

Note that if $\alpha \in CC(A_{r+1})$ then $\alpha \notin CC(A_j)$ for $j \neq r+1$, and if $\alpha \in CC(C_r)$ then $\alpha \notin CC(C_j)$ for $j \neq r$. Now the proof follows as Lemma 3.9. $\square$

Now we may continue the proof of Theorem 4.4. Recall that we use induction on m, the number of components in the minimal primary decompositon.

$m=2$. Let $I = I_1 \cap I_2$. By using the Mayer–Vietoris sequence we get the following cases:

1. $H^{h_1}_I(R) \cong H^{h_1}_{I_1}(R)$,
   $H^{h_2}_I(R) \cong H^{h_2}_{I_2}(R)$ and
   $H^{h_{12}-1}_I(R) \cong H^{h_{12}}_{I_1+I_2}(R)$.
2. $H^{h_1}_I(R) \cong H^{h_1}_{I_1}(R) \oplus H^{h_1}_{I_2}(R)$ and
   $H^{h_{12}-1}_I(R) \cong H^{h_{12}}_{I_1+I_2}(R)$.
3. $0 \to H^{h_1}_{I_1}(R) \to H^{h_1}_I(R) \to H^{h_1+1}_{I_1+I_2}(R) \to 0$ and
   $H^{h_2}_I(R) \cong H^{h_2}_{I_2}(R)$.
4. $0 \to H^{h_1}_{I_1}(R) \oplus H^{h_1}_{I_2}(R) \to H^{h_1}_I(R) \to H^{h_1+1}_{I_1+I_2}(R) \to 0$.

Applying the exact sequence of local cohomology to each case we obtain

1. $H^p_{\mathfrak{m}}(H^{h_1}_I(R)) \cong H^p_{\mathfrak{m}}(H^{h_1}_{I_1}(R))$,
   $H^p_{\mathfrak{m}}(H^{h_2}_I(R)) \cong H^p_{\mathfrak{m}}(H^{h_2}_{I_2}(R))$ and
   $H^p_{\mathfrak{m}}(H^{h_{12}-1}_I(R)) \cong H^p_{\mathfrak{m}}(H^{h_{12}}_{I_1+I_2}(R))$.
2. $H^p_{\mathfrak{m}}(H^{h_1}_I(R)) \cong H^p_{\mathfrak{m}}(H^{h_1}_{I_1}(R)) \oplus H^p_{\mathfrak{m}}(H^{h_1}_{I_2}(R))$ and
   $H^p_{\mathfrak{m}}(H^{h_{12}-1}_I(R)) \cong H^p_{\mathfrak{m}}(H^{h_{12}}_{I_1+I_2}(R))$.
3. $0 \to H^{n-h_1-1}_{\mathfrak{m}}(H^{h_1}_I(R)) \to E_R(R/\mathfrak{m}) \to E_R(R/\mathfrak{m}) \to H^{n-h_1}_{\mathfrak{m}}(H^{h_1+1}_{I_1+I_2}(R)) \to 0$ and
   $H^p_{\mathfrak{m}}(H^{h_2}_I(R)) \cong H^p_{\mathfrak{m}}(H^{h_2}_{I_2}(R))$.

We want to see that $H^{n-h_1-1}_{\mathfrak{m}}(H^{h_1}_I(R))$ and $H^{n-h_1}_{\mathfrak{m}}(H^{h_1+1}_{I_1+I_2}(R))$ vanish. We have $CC(H^{h_1}_I(R)) = CC(H^{h_1}_{I_1}(R)) + CC(H^{h_1+1}_{I_1+I_2}(R))$. We can suppose $h_1+1=n$, so

$$CC(H^{n-1}_I(R)) = T^*_{\{x_1=\cdots=x_{n-1}=0\}}X + T^*_{\{x_1=\cdots=x_n=0\}}X.$$

Denote $M = H^{n-1}_I(R)$. Using the Čech complex

$$0 \to M \to \bigoplus M_{f_i} \to \bigoplus M_{f_i f_j} \to \cdots$$

and the result of [4], see Section 2.1.2, we get

$$CC(M_{x_n}) = T^*_{\{x_1=\cdots=x_{n-1}=0\}}X + T^*_{\{x_1=\cdots=x_n=0\}}X$$

and the characteristic cycle of the other localizations vanishes. Therefore, the complex reduces to

$$0 \to M \to M_{x_n} \to 0$$

and so $H^p_{\mathfrak{m}}(H^{h_1}_I(R)) = 0\ \forall p$.

4. $0 \to H^{n-h_1-1}_{\mathfrak{m}}(H^{h_1}_I(R) \to E(R/\mathfrak{m}) \to E(R/\mathfrak{m})^2 \to H^{n-h_1}_{\mathfrak{m}}(H^{h_1+1}_{I_1+I_2}(R)) \to 0$.

We want to see that $H^{n-h_1-1}_{\mathfrak{m}}(H^{h_1}_I(R)$ vanishes and $CC(H^{n-h_1}_{\mathfrak{m}}(H^{h_1+1}_{I_1+I_2}(R))) = T^*_{\{x_1=\cdots=x_n=0\}}X$. As above, we can suppose $h_1+1=n$. Then we have

$$CC(H^{n-1}_I(R)) = T^*_{\{x_1=\cdots=x_{n-1}=0\}}X + T^*_{\{x_1=\cdots=x_{n-2}=x_n=0\}}X \\ +T^*_{\{x_1=\cdots=x_n=0\}}X.$$

Denote $M = H^{n-1}_I(R)$. The Čech complex reduces to

$$0 \to M \to M_{x_{n-1}} \oplus M_{x_n} \to 0$$

with

$$CC(M_{x_{n-1}} \oplus M_{x_n}) = T^*_{\{x_1=\cdots=x_{n-1}=0\}}X + T^*_{\{x_1=\cdots=x_{n-2}=x_n=0\}}X \\ +2T^*_{\{x_1=\cdots=x_n=0\}}X.$$

Therefore we get the desired result.

$m>2$. Consider the Mayer–Vietoris sequence with

$$U = I_1 \cap \cdots \cap I_{m-1},$$
$$V = I_m,$$
$$I = U \cap V = (I_1 \cap \cdots \cap I_{m-1}) \cap (I_m),$$
$$U + V = I_1 \cap \cdots \cap I_{m-1} + I_m.$$

Then we have

$$0 \to C_r \to H^r_{U\cap V}(R) \to A_{r+1} \to 0.$$

Recall that $CC(C_r) \subseteq CC(H^r_U(R)) + CC(H^r_V(R))$ and $CC(A_{r+1}) \subseteq CC(H^{r+1}_{U+V}(R))$. By induction and Lemma 4.5 we have

$$CC(H^r_U(R)) = \sum_{I_i+I_m\in\Omega_{2,r}} CC(H^r_{I_i+I_m}(R)) + \cdots + \sum_{I_1+\cdots+I_m\in\Omega_{m,r}} CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-1}}(R)),$$
$$CC(H^r_V(R)) = CC(H^r_{I_{m-1}+I_m}(R)),$$

$$CC(H^{r+1}_{U+V}(R)) = \sum_{I_i+I_m\in\Omega^m_{2,r+1}} CC(H^{r+1}_{I_i+I_m}(R)) + \cdots$$
$$+ \sum_{I_1+\cdots+I_m\in\Omega^m_{m,r+1}} CC(H^{r+(m-2)}_{I_1+\cdots+I_m}(R)).$$

To get the desired formula we only need to describe $CC(Z_p)$, so it is enough to prove the following:

Claim.

$$CC(Z_p) = \sum T^*_{\{x_1=\cdots=x_n=0\}}X,$$

where the sum is taken over the cycles

$$T^*_{\{x_1=\cdots=x_n=0\}}X = CC(H^p_{\mathfrak{m}}(H^{r+j}_{I_{i_1}+\cdots+I_{i_j}+I_m}(R))) \in CC(H^p_{\mathfrak{m}}(A_{r+1})),$$

such that

$$T^*_{\{x_1=\cdots=x_n=0\}}X = CC(H^{p+1}_{\mathfrak{m}}(H^{r+j-1}_{I_{i_1}+\cdots+I_{i_j}}(R))) \in CC(H^{p+1}_{\mathfrak{m}}(C_r)).$$

The proof of the claim is as above.

5. Examples

Let $R=k[x,y,z,t]$ or $R=k[[x,y,z,t]]$ and $\mathfrak{m}=(x,y,z,t)$. Let $I=(xyz,xyt,xzt)$. Its minimal primary decomposition is $I=(x)\cap(y,z)\cap(z,t)\cap(y,t)$ and the set ω of sums of face ideals is

$$\omega_1 = \begin{Bmatrix} I_1=(x) \\ I_2=(y,z) \\ I_3=(z,t) \\ I_4=(y,t) \end{Bmatrix}, \quad \omega_2 = \begin{Bmatrix} I_1+I_2=(x,y,z) \\ I_1+I_3=(x,z,t) \\ I_2+I_3=(y,z,t) \\ I_1+I_4=(x,y,t) \\ I_2+I_4=(y,z,t) \\ I_3+I_4=(y,z,t) \end{Bmatrix},$$

$$\omega_3 = \begin{Bmatrix} I_1+I_2+I_3=\mathfrak{m} \\ I_1+I_2+I_4=\mathfrak{m} \\ I_1+I_3+I_4=\mathfrak{m} \\ I_2+I_3+I_4=(y,z,t) \end{Bmatrix}, \quad \omega_4 = \{I_1+I_2+I_3+I_4=\mathfrak{m}\}.$$

Applying Algorithm 1 and collecting the sums of face ideals by the number of summands and height we get

$$\Omega_{1,1} = \{I_1\},$$
$$\Omega_{1,2} = \{I_2, I_3, I_4\},$$

$$\Omega_{2,2} = \{I_1 + I_2, I_1 + I_3, I_2 + I_3, I_1 + I_4, I_3 + I_4\},$$
$$\Omega_{3,2} = \{I_1 + I_2 + I_3, I_1 + I_2 + I_4\}.$$

Therefore, we can describe the characteristic cycle of the local cohomology modules supported on I. Namely,

$$\begin{aligned} CC(H_I^1(R)) &= CC(H_{I_1}^1(R)). \\ CC(H_I^2(R)) &= CC(H_{I_2}^2(R)) + CC(H_{I_3}^2(R)) + CC(H_{I_4}^2(R)) \\ &\quad + CC(H_{I_1+I_2}^3(R)) + CC(H_{I_1+I_3}^3(R)) + CC(H_{I_2+I_3}^3(R)) \\ &\quad + CC(H_{I_1+I_4}^3(R)) + CC(H_{I_3+I_4}^3(R)) + CC(H_{I_1+I_2+I_3}^4(R)) \\ &\quad + CC(H_{I_1+I_2+I_4}^4(R)). \end{aligned}$$

In terms of conormal bundles relative to a subvariety we have

$$\begin{aligned} CC(H_I^1(R)) &= T_{\{x=0\}}^* X, \\ CC(H_I^2(R)) &= T_{\{y=z=0\}}^* X + T_{\{z=t=0\}}^* X + T_{\{y=t=0\}}^* X \\ &\quad + T_{\{x=y=z=\}}^* X + T_{\{x=z=t=0\}}^* X + T_{\{x=y=t=0\}}^* X + 2T_{\{y=z=t=0\}}^* X \\ &\quad + 2T_{\{x=y=z=t=0\}}^* X. \end{aligned}$$

Note that there are two local cohomology modules different from zero, so R/I is not Cohen–Macaulay by Proposition 3.1.

Applying Algorithm 2 we get

$$\Gamma_1 = \Gamma_{1,1} = \{I_1\}, \quad \Gamma_2 = \{\emptyset\}, \quad \Gamma_3 = \{\emptyset\}, \quad \Gamma_4 = \{\emptyset\}.$$

Therefore, we can compute the Lyubeznik numbers of R/I. Since $ht\, I_1 = 1$ we have $CC(H_{\mathfrak{m}}^3(H_I^1(R))) = T_{\{x=y=z=t=0\}}^* X$. Therefore $\lambda_{3,3}(R/I) = 1$ and all the other Lyubeznik numbers vanish. The type is then

$$\Lambda(R/I) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 1 \end{pmatrix}.$$

Remark 5.1. Lyubeznik asked [12, Question 4.5] if $\lambda_{d,d}(R/I) = 1$ for all R/I where $d = dim\, R/I$. Walther [16] gave a negative answer when $d = 2$.

We may give counterexamples for any dimension d as follows:

Consider $I = I_1 \cap \cdots \cap I_m$ such that $h_i > 1\ \forall i$ and $ht(I_{i_1} + \cdots + I_{i_s}) = ht\, I_{i_1} + \cdots + ht\, I_{i_s}\ \forall s$. Then all the sum of face ideals are non-paired and non-almost paired, so $\lambda_{d,d}(R/I)$ is the number of face ideals in the minimal primary descomposition of height $n - d$.

Example. Let $R=k[[x_1,x_2,x_3,x_4,x_5,x_6,x_7]]$ and $I=(x_1,x_2)\cap(x_3,x_4)\cap(x_5,x_6,x_7)$. Then the type of R/I is

$$\Lambda(R/I)=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 2 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 2 \end{pmatrix}.$$

6. Combinatorics of the Stanley–Reisner ring and multiplicities

Let $R=k[x_1,\dots,x_n]$ be the polynomial ring in n variables over a field k of characteristic zero. We want to study the combinatorics of the Stanley–Reisner ring R/I, where $I\subseteq R$ is a squarefree monomial ideal.

Consider the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$.

$$CC(H_I^{n-i}(R))=\sum m_j T^*_{Y_j}X.$$

Then we define

$$\gamma_{p,i}:=\left\{\sum m_j : dim\, Y_j=p\right\}.$$

Let $d=dim\, R/I$. Then by Theorem 3.8 we get the same properties as Lyubeznik numbers:

(i) $\gamma_{p,i}=0$ if $i>d$.
(ii) $\gamma_{p,i}=0$ if $p>i$.
(iii) $\gamma_{d,d}\neq 0$.

So we can also consider the tringular matrix given by $\gamma_{p,i}$

$$\begin{pmatrix} \gamma_{0,0} & \cdots & \gamma_{0,d} \\ & \ddots & \vdots \\ & & \gamma_{d,d} \end{pmatrix}.$$

Remark 6.1. We have the following:
(a) $H_I^{n-i}(R)=0$ if and only if $\gamma_{p,i}=0\ \forall p$.
(b) R/I is Cohen–Macaulay if and only if $\gamma_{p,i}=0\ \forall i\neq d,\ \forall p$.
(c) $dim\,\mathrm{Supp}\, H_I^{n-i}(R)\leq i$.

Consider the simplicial complex Δ associated to the Stanley–Reisner ring R/I and the face ideals appearing in the minimal primary decomposition of I. Recall that the sums of face ideals corresponds to the intersection of maximal faces of Δ. Then, by an easy computation, we can describe the f-vector and the h-vector of Δ in terms of $\gamma_{p,i}$.

For this purpose we introduce

$$\mathscr{B}_k := \gamma_{k,k} - \gamma_{k,k+1} + \cdots + (-1)^{d-k}\gamma_{k,d}.$$

Proposition 6.2. *The f-vector $f(\Delta)=(f_{-1},\ldots,f_{d-1})$, where f_k is the number of k-dimensional faces of Δ, is of the form*

$$f_k = \binom{d}{k+1}\mathscr{B}_d + \binom{d-1}{k+1}\mathscr{B}_{d-1} + \cdots + \binom{k+1}{k+1}\mathscr{B}_{k+1}.$$

Remark 6.3. We can also give the following:
(a) $1 = f_{-1} = \mathscr{B}_d + \cdots + \mathscr{B}_0 = \sum(-1)^{p+i}\gamma_{p,i}$.
(b) The Euler characteristic of Δ is $\chi(\Delta) = 1 - \mathscr{B}_0$.

Proposition 6.4. *The h-vector $h(\Delta)=(h_0,h_1,\ldots)$, given by the relation*

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^i$$

is of the form

$$h_k = (-1)^k\left(\binom{d}{k}\mathscr{B}_0 + \binom{d-1}{k}\mathscr{B}_1 + \cdots + \binom{k}{k}\mathscr{B}_{d-k}\right).$$

Remark 6.5. It is easy to prove that the multiplicities of the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$ are invariants of R/I for any ideal $I \subseteq R$. In the case of squarefree monomial ideals we can see that the invariants $\gamma_{p,i}$ are finer than the f-vector or the h-vector. They are equivalent when R/I is Cohen–Macaulay.

Example. Let $R = k[x_1,x_2,x_3,x_4,x_5]$. Consider

$$I_1 = (x_1,x_3)\cap(x_1,x_4)\cap(x_1,x_5)\cap(x_2,x_3)\cap(x_2,x_4).$$

By Theorem 3.8 we get the matrix of $\gamma_{p,i}$-invariants

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 2 \\ & & 0 & 6 \\ & & & 5 \end{pmatrix}.$$

The f-vector is $f(\Delta_1) = (1,5,9,5)$

$$I_2 = (x_1,x_4)\cap(x_1,x_5)\cap(x_2,x_5)\cap(x_3,x_5)\cap(x_4,x_5)\cap(x_1,x_2,x_3).$$

The matrix of $\gamma_{p,i}$-invariants is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 1 & 3 \\ & & 1 & 7 \\ & & & 5 \end{pmatrix}.$$

The f-vector is also $f(\Delta_2) = (1,5,9,5)$

So we can obtain examples of simplicial complexes with the same f-vector and different $\gamma_{p,i}$-invariants. Note that R/I_1 is Cohen–Macaulay but R/I_2 is not Cohen–Macaulay.

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